The maptree: A fine-grained formal representation of space

Michael Worboys

School of Computing and Information Science
University of Maine, Orono ME 04473, USA
http://www.worboys.org

Abstract. This paper introduces a new formal structure, called the maptree, that is shown to uniquely specify, up to homeomorphism, the topological structure of embeddings of graphs in orientable, closed surfaces. A simple modification is made to show that the representation also works for planar embeddings. It is shown that the maptrees are capable of providing a rich representation of the topology of 2D spatial objects and their relationships. The maptree representation is then used to characterize some properties of topological change in these embeddings.

Keywords: maptree, topology, topological change, geographic information science theory

1 Introduction

Although there is a considerable body of research on topological relationships between spatial objects (see, for example, [6, 3]), there is much less that takes into account the finer topological details of these objects and their relationships. For example, in the region connection calculus, connection between regions at a point, at several points, at a line, or at several lines, cannot be distinguished by the basic theory. Similarly the theory of topological change developed by the author and colleagues in [5, 4] presents a theory of topological change that can distinguish, for example, between a hole emerging from a point in the center of a region and the same region merging with itself to create a hole, but cannot distinguish a merge of two regions at a point, or at a linear boundary.

The purpose of this paper is to describe research that brings together two separate formal descriptions of topological configurations, namely combinatorial maps [2, 8] and adjacency trees [1, 7] to provide a fine-grained representation of spatial objects and their relationships. The structure presented in this paper is an edge-labelled, node-colored tree, that we name a *maptree* The spatial objects under consideration are configurations of regions, and it is convenient to consider such configurations as embeddings of graphs in orientable closed surfaces or the plane.

A *graph* is defined in the usual way as a set of vertices and edges between vertices. In this paper we allow *loops*, that is edges that connect a vertex to itself, and also multiple edges between two vertices. Often, such structures are referred

to as *pseudographs*, but we continue to use the term "graph" throughout this paper. A graph is *connected* if any pair of its vertices may be linked by a chain of adjacent edges.

Informally, an *embedding* of a graph in a surface is a drawing of the graph on the surface in such a way that its edges may intersect only at their endpoints. The surfaces of interest here are the orientable closed surfaces in $\Re^n$, and it is well known (Möbius classification theorem for orientable, closed surfaces, 1863) that such a surface is homeomorphic to a sphere with g handles (g-holed torus), for $g \geq 0$. The non-negative integer g is referred to as the *genus* of the surface. We shall usually only be concerned with the sphere (of genus zero). As well as these closed surfaces we also shall consider the Euclidean plane $\Re^2$ (not a closed surface). From now on, we shall assume all surfaces under consideration are orientable, and omit that term from their descriptors.

Graph embeddings in closed surfaces have the property that the complement in the surface of an embedding of a connected graph is a collection of regions or *faces*, and each of these faces is a 2-manifold. If, furthermore, each of the faces is homeomorphic to a disc, the embedding is called a *2-cell embedding*. When the graph is embedded in the Euclidean plane, then one of the faces will be of infinite extent, and called the *external face*. Of course, not all graphs can be embedded in the plane.

2 Permutations and combinatorial maps

In this section we review the basic concepts around the combinatorial map. Such a map provides a unique (up to homeomorphism) symbolic representation of an embedding of a connected graph. By way of introduction, we review some material on permutations.

Let $A = \{a, b, \ldots, k\}$ be a finite collection of elements. We call any bijective function $\phi : A \to A$ a *permutation* of A. Essentially, we can think of ϕ as rearranging the elements of A. Now, any permutation can always be written as a collection of cycles $(a_1 a_2 \ldots a_n)$, where $a_2 = \phi a_1$, $a_3 = \phi a_2$, and so on, and $a_1 = \phi a_n$. So, for example, suppose $A = \{a, b, c, d, e\}$, and $b = \phi a$, $c = \phi b$, $a = \phi c$, $e = \phi d$, and $d = \phi e$. Then ϕ may be written in cycle notation as $\phi = (abc)(de)$.

Suppose now that we have a collection of permutations of A, $\Phi = \phi_1, \ldots, \phi_m$. Then Φ is *transitive* if, given any elements $x, y \in A$, we can transform x to y by a sequence of permutations from Φ. That is,

$$x \xrightarrow{\phi_{i_1}} x_1 \xrightarrow{\phi_{i_1}} \ldots x_p \xrightarrow{\phi_{i_p}} y$$

We have now sufficient preliminaries to define a combinatorial map.

Definition 1 *A* combinatorial map $\boldsymbol{M}\langle S, \alpha, \tau\rangle$ *consists of:*

1. *A finite set S of elements, called* semi-edges*, where the number of semi-edges is even. We can write S as $S = \{a, \overline{a}, b, \overline{b}, \ldots, k, \overline{k}\}$*
2. *A permutation α of S*

3. A permutation τ of S, which in cyclic form is $\tau = (a\overline{a})(b\overline{b})\ldots(k\overline{k})$

subject to the constraint that $\{\tau, \alpha\}$ is transitive.

An example of a combinatorial map that we shall use again is given by $\mathbf{M}_1\langle S, \alpha, \tau\rangle$, where $S = \{a, \overline{a}, b, \overline{b}, c, \overline{c}\}$, $\alpha = (\overline{a}c\overline{b}\overline{c})(a)(b)$, and $\tau = (a\overline{a})(b\overline{b})(c\overline{c})$. It is easy to check that $\{\tau, \alpha\}$ is transitive.

We are now able to relate combinatorial maps to graph embeddings using the following key result due to Edmonds [2] and Tutte [8].

Theorem 1 *(Edmunds, Tutte) Each combinatorial map provides a topologically unique (up to homeomorphism of the surficial embeddings) representation of a 2-cell graph embedding in a closed surface. Conversely, every 2-cell graph embedding in a closed surface can be uniquely (up to permutation group isomorphism) be represented by a combinatorial map.*

We do not repeat a formal proof of this result, but indicate the construction. Given a combinatorial map $\mathbf{M}\langle S, \alpha, \tau\rangle$, the 2-cell embedding is constructed as follows. Each edge of the embedded graph is represented by a pair of semi-edges, called a *facing pair*, transposed by τ. Each cycle of α defines the ordering of semi-edges around each face of the embedding. Each face is defined as the region on the left while traversing the semi-edges of a cycle of α. We may note that the constituent cycles of α are sufficient to uniquely reconstruct the embedding. The constituent cycles of α are termed the *α-cycles* of $\mathbf{M}$.

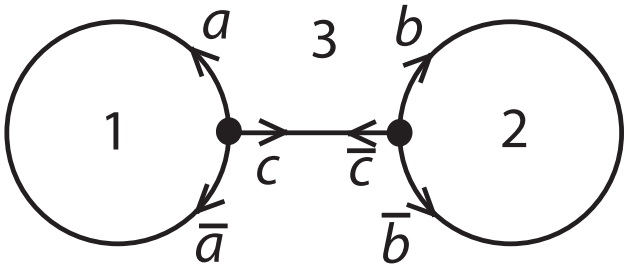

Fig. 1. Embedding of $\mathbf{M}_1$ in the sphere

Figure 1 illustrates this construction using combinatorial map, $\mathbf{M}_1\langle S, \alpha, \tau\rangle$. We observe that the three α-cycles $(\overline{a}c\overline{b}\overline{c})$,$(a)$, and (b) define the three faces, labelled 1,2, and 3. For example, the face labeled 1 is defined as the region to the left while traveling the semi-edge a, while the "outer" face labeled 3 is the region to the left while traversing the directed path given by the cycle of semi-edges $\overline{a} \to c \to b \to \overline{c} \to \overline{a}$.

We note that although the configuration is of necessity reproduced on the plane paper, it is actually a spherical embedding. This can be seen by observing

figure 2, where we show two homeomorphic embeddings of $\mathbf{M}_1\langle S, \alpha, \tau\rangle$ in a sphere. Note also, for reason of clarity, we show only one of each pair of semi-edges.

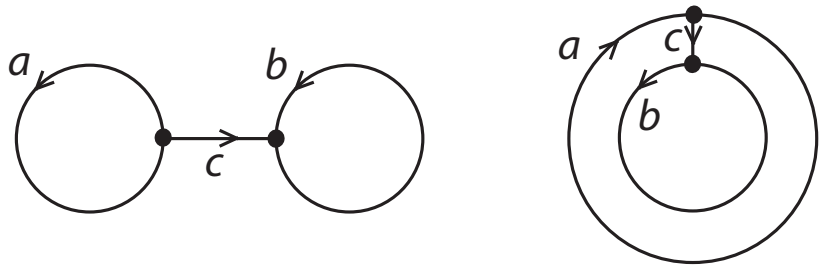

Fig. 2. Two homeomorphic embeddings of $\mathbf{M}_1$ in the sphere

We now focus on the nature of the closed surface in which the configuration represented by a combinatorial map is embedded. This surface must be unique by Theorem 1. To determine the surface from the combinatorial map, we need to determine the number of vertices of the embedded graph. To do this, we calculate a further permutation β by the formula $\beta = \tau\alpha^{-1}$ where the product is a composition of functions, and α^{-1} denotes the inverse function of α. In our example $\beta = (a\bar{a}c)(b\bar{b}\bar{c})$. Now we note that each cycle of β represents a vertex, and the cycle itself represents the ordering of semi-edges around the vertex.

Now we invoke the famous result of Euler and Poincaré:

Theorem 2 *(Euler-Poincaré) Given a 2-cell embedding of a graph in a surface of genus g. Suppose that V, E, and F are the numbers of the embedding's vertices, edges, and faces, respectively. Then:*

$$V - E + F = 2 - 2g$$

Given a combinatorial map, $\mathbf{M}\langle S, \alpha, \tau\rangle$, the genus of it's embedding surface can then be calculated as follows.

$$\begin{aligned} V &= \text{the number of constituent cycles of } \beta \\ E &= \text{the number of constituent cycles of } \tau \\ F &= \text{the number of constituent cycles of} \alpha \end{aligned}$$

and then, by Theorem , the genus g is given by $V-E+F = 2-2g$. We sometimes refer to the genus of the combinatorial map, meaning the genus of its embedding surface.

For our example, $V = 2, E = 3, F = 3$, and so $2 - 2g = 2$, and $g = 0$ which accords with our knowledge that the embedding surface is a sphere.

2.1 Planar and spherical 2-cell embeddings

For practical purposes, the embedding surfaces of interest are the sphere (closed surface of genus zero) and the Euclidean plane. The above results on combinatorial maps apply to closed surfaces, and therefore do not apply directly to the Euclidean plane. However, any graph that is embeddable in the sphere is embeddable in the plane, and conversely. The Euclidean plane is topologically equivalent to a punctured sphere. Suppose we have a map that is 2-cell embeddable on the surface of a sphere. We now puncture the sphere, taking care that the puncture does not lie on an edge or vertex of the embedding. (The cases where the puncture lies on an edge or vertex does not concern us, as we are not concerned with unbounded embeddings in the plane.) We now have a planar embedding of the map, and the face in which the puncture occurs is the infinite external face. Of course, the topological nature of a planar 2-cell embedding is dependent upon in which face of the spherical 2-cell embedding the puncture occurs. The plane places an extra piece of structure on the 2-cell embedding in that we have the notion of the infinite face. So, a 2-cell embedding in the sphere may correspond to many topologically distinct 2-cell embeddings in the plane. We can see this by re-examining figure 2, where the left and right embeddings are the same on the sphere but distinct on the plane.

In order that a combinatorial map can uniquely specify a planar 2-cell embedding, all we need to do is specify which cycle represents the boundary of the external face. We indicate the distinguished external boundary cycle in α by square brackets. In our example, the lefthand and righthand planar 2-cycle embeddings are given by $[\overline{a}c\overline{b}\overline{c}](a)(b)$ and $\alpha = (\overline{a}c\overline{b}\overline{c})[a](b)$, respectively. We can then invoke a slightly extended version of Theorem 1(Edmunds, Tutte) to guarantee topological uniqueness of the representation.

3 Representations of non-connected graph embeddings

Up to now, the focus has been on connected graphs, because only connected graphs have 2-cell embeddings and are representable by combinatorial maps. In this section we extend the algebraic representation of graph embeddings as combinatorial maps to also represent embeddings of non-connected graphs. In our examples, we focus on the Euclidean plane and sphere, as these are the important cases in practice. However, our results carry over to surfaces of any genus.

When we remove the constraint that the graphs are connected, we have to add some extra structure to the algebraic representation. To see this, consider the two graphs, each embedded in the surface of a sphere, shown on the lefthand and righthand sides of figure 3. Both embeddings would have the same representation as a collection of three combinatorial maps. However, it is easy to see that these embeddings are not topologically equivalent, in the sense that there does not exist a homeomorphism of the sphere that maps one embedding to the other. How graph embeddings stand with respect to one another becomes an issue, and we cannot just represent the embedding of a non-connected graph as a set of maps.

In this section we develop the extra structure needed to provide topologically unique representations for planar and spherical embeddings of non-connected graphs.

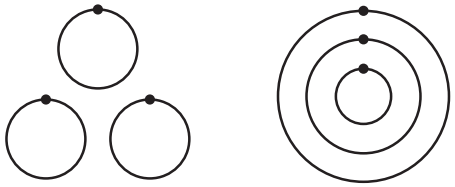

Fig. 3. Two non-homeomorphic embeddings in the sphere

As a preliminary to our main result, we revisit a known representation of a collection of closed curves embedded in a surface as a tree.

3.1 Regions created from nested closed curves

Consider a closed surface in which is embedded a disjoint collection of closed curves. An example of such a configuration is shown in the left-hand part of figure 4. Given any such configuration, we may define an associated tree as follows:

Definition 2 *Suppose given a disjoint collection* $\boldsymbol{C}$ *of closed curves embedded in a closed surface, such that each closed curve viewed on its own results in a two-cell embedding in the surface. Such a collection of curves partitions the surface into a set of regions. We define the* adjacency tree *of* $\boldsymbol{C}$ *as a graph* $\boldsymbol{T_C} = \langle N, E \rangle$*, where* N *is a set of nodes, each node representing one of the regions and* E *is a set of edges. Two nodes are joined by an edge if their associated regions share a common boundary.*

Note that this definition begs the question whether or not $\mathbf{T_C}$ is a tree. To see this, consider the number of nodes and edges in the graph. The smallest possible such graph has no edges, and a single node, representing the case where the region covers the entire surface. In the general case, if we add another closed curve (edge) to our configuration, because of the two-cell embedding property of the curve, adding the curve adds one more region (node) to $\mathbf{T_C}$. So, by induction, $\mathbf{T_C}$ has one more node than edges. Because $\mathbf{T_C}$ must be connected, it follows that $\mathbf{T_C}$ is a tree.

It is known that that an adjacency tree characterizes the configuration of regions in the plane uniquely, up to homeomorphism from the surface to itself (see, for example, [1, 7]).

Note that if the embedding is planar, then we can distinguish the infinite external region by making the node representing it in the adjacency tree the root of a rooted tree. So, in the example in figure 4, the node labelled 1 will be the root.

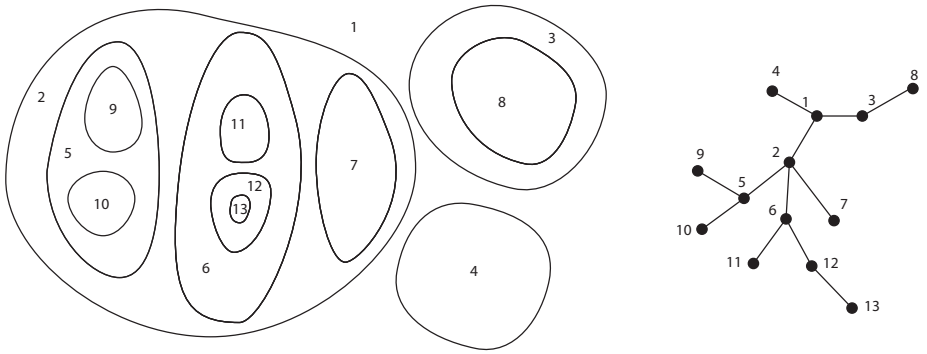

Fig. 4. A collection of nested closed curves and its tree

3.2 Maptrees

The key insight of research reported in this paper is that a combinatorial map provides a symbolic representation of the 2-cell embedding of a connected graph in a closed surface (a connected, complex structure), while an adjacency map provides a symbolic representation of a collection of closed curves embedded in a closed surface (a non-connected, simple structure). We now show how these constructions can be combined.

Definition 3 *A* bw-tree $\boldsymbol{X}$ *is a colored tree with the nodes colored black or white, respectively, subject to the condition that no two adjacent nodes have the same color. A bw-tree is called a* star *if it contains exactly one black node.*

So as to more easily follow the construction that follows, we illustrate with the embedding shown in figure 5. We assume to begin with that the embedding surface is a sphere. We have omitted the directions of arcs for simplicity.

As a first step, we use a bw-tree to represent the nesting properties of the components of the graph. Figure 6 shows this for our example. The components, labelled $M_1, \ldots, M_6$ in the lefthand side of the figure can each be conceived as a black region, represented by the black nodes in the bw-tree on the righthand side of the figure. The regions labelled $1, \ldots, 9$ delineated by the edges of the graph are represented by white nodes in the bw-tree.

This bw-tree uniquely represents how the components and regions stand in the relation to each other, but does not provide details about the topology of the components themselves. To provide this information, we need to consider the combinatorial maps of the components. The graph has six components, and each embedded component $M_1, \ldots, M_6$ considered alone is a 2-cell embedding and so can be represented by a map (using the permutation α to represent each map), as follows:

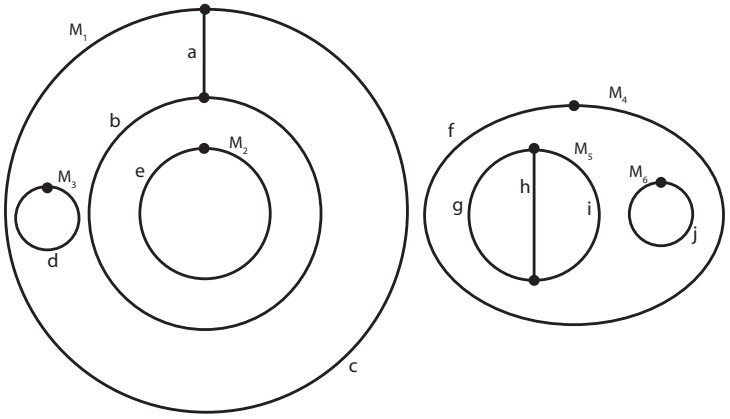

Fig. 5. Embedding of a non-connected graph

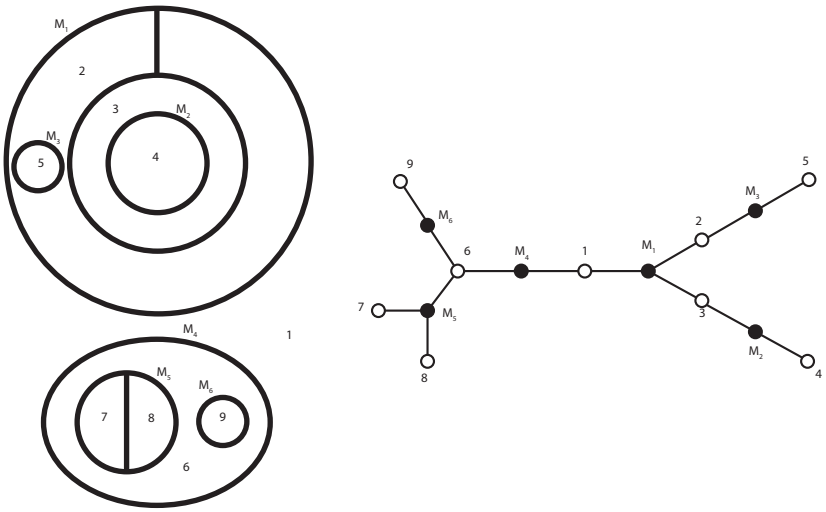

Fig. 6. Region-oriented view of the embedding in figure 5

$$\mathbf{M}_1 = (a\overline{ca}b)(\overline{b})(c)$$
$$\mathbf{M}_2 = (e)(\overline{e})$$
$$\mathbf{M}_3 = (d)(\overline{d})$$
$$\mathbf{M}_4 = (f)(\overline{f})$$
$$\mathbf{M}_5 = (gi)(\overline{g}h)(\overline{hi})$$
$$\mathbf{M}_6 = (j)(\overline{j})$$

The final stage of the construction is shown in figure 7. Each of the components is represented by a star, labelled with α-cycles from its combinatorial map. The stars are joined together in such a way that the edges connecting a white node form the boundary of the region represented by that node. This structure we term a *maptree*.

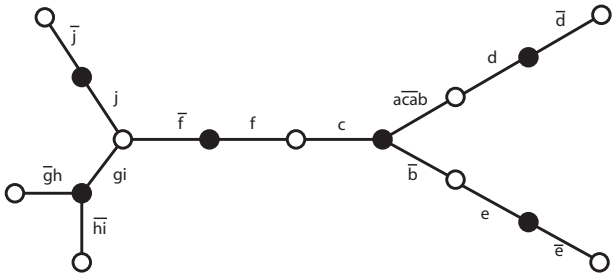

Fig. 7. Maptree of the embedding in figure 5

We are now ready to give the formal definition of a maptree.

Definition 4 *Let $\mathcal{M}$ be a finite collection of combinatorial maps. A* maptree *is an edge-labelled bw-tree $\mathcal{T}_{\mathcal{M}}$ such that the edges of each star are labelled by the α-cycles of one of the combinatorial maps in $\mathcal{M}$.*

It is clear from the above example how we should interpret a maptree as a collection of 2-cell embeddings. However, let us just spell out the details. The interpretation of maptree $\mathcal{T}_{\mathcal{M}}$ is that each black node of $\mathcal{T}_{\mathcal{M}}$ represents the 2-cell embedding of the map associated with that node. If two black nodes are connected via a white node, then the two cycles labeling the two edges joining the nodes represent the regions that "face up to each other" in the embedding.

We have the following proposition.

Proposition 1 *Let $\mathcal{M}$ be a finite collection of combinatorial maps of genus zero, and $\mathcal{T}_{\mathcal{M}}$ be a maptree. Then $\mathcal{T}_{\mathcal{M}}$ provides a unique representation (up*

to homeomorphism of the sphere) of the non-connected graph embedding of the connected components represented by $\mathcal{M}$.

We sketch the proof of this result by indicating the steps in the construction of the graph embedding from a map tree, and note that there are no topological 'degrees of freedom' in the process. The process is essentially the reverse of the process described above, where the maptree is constructed from a graph embedding.

Suppose given a maptree $\mathcal{T}_{\mathcal{M}}$, where $\mathcal{M}$ is a finite collection of combinatorial maps.

Step 1: Consider $\mathcal{T}_{\mathcal{M}}$ as specifying the topological configuration of black and white regions on the sphere. As already discussed, such a configuration is unique up to homeomorphism of the sphere.

Step 2: For each black region consider the node n of $\mathcal{T}_{\mathcal{M}}$ that represents it. Then, there is some map $M \in \mathcal{M}$ such that n is said to be the node associated with **M**. Replace this region with the graph embedding represented by **M**. This replacement also results in a topologically unique embedding.

Step 3: The resulting graph embedding, formed as the composite of all the embeddings in Step 2, is the result, and is unique up to homeomorphism of the sphere.

The proposition can be generalized to the case where the maps in $\mathcal{M}$ embedded into surfaces of any genus. In that case, the genus of the embedding surface of the maptree is the sum of the genera of the constituent combinatorial maps.

3.3 Maptrees for planar embeddings

The maptree construction is based on the notions of adjacency trees and combinatorial maps, both of which work for embeddings in closed surfaces. Many practical cases are concerned with embeddings in the plane. In the same way as combinatorial maps are modified, we can modify the map tree construction to account for planar embeddings. As before, we need to distinguish the infinite exterior region, represented by one of the white nodes. We do this by making this the root of a rooted tree. The formal definition follows.

Definition 5 *Let $\mathcal{M}$ be a finite collection of combinatorial maps. A* planar maptree *is an edge-labelled rooted bw-tree $\mathcal{T}_{\mathcal{M}}$ such that the edges of each star are labelled by the α-cycles of one of the combinatorial maps in $\mathcal{M}$, and the root of the tree is a white node.*

As an example, the planar maptree for the planar embedding shown in figure 5 is shown in figure 8, where the root is at the top of the figure.

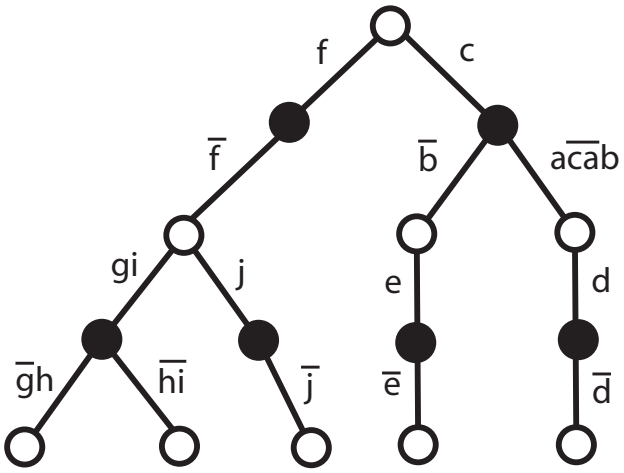

Fig. 8. Maptree of the planar embedding in figure 5

4 Maptrees and topological change

The previous sections have shown how the maptree construct provides a topologically complete representation of configurations of regions in a surface. In this section we provide a few examples that demonstrate how maptree operations serve to represent varieties of topological change. We assume that the embedding surfaces is the plane. However, all the results carry forward to any closed surface. The examples are illustrative of general results, not proved here, but the important concepts are illustrated.

4.1 Merge of two regions at a point

The top portion of figure 9 shows the merger of two regions at a common point, while the bottom of the figure shows the corresponding maptree transformation. We can note that the merger is represented by a folding together of the edges labeled $\overline{a}$ and $\overline{b}$. This is a general principle that holds for more complex examples – a point merger is represented by a folding about a white node, thus merging two black nodes. The labels of the components of the fold are concatenated. In general, where the labeling is more than a single edge, there are many ways of performing the concatenation of cycles, each corresponding to a distinct point merge.

4.2 Merge of two regions at an edge

The first transition of the top portion of figure 10 shows the merger of two regions using the region edges a and c. Edges a and c merge together forming

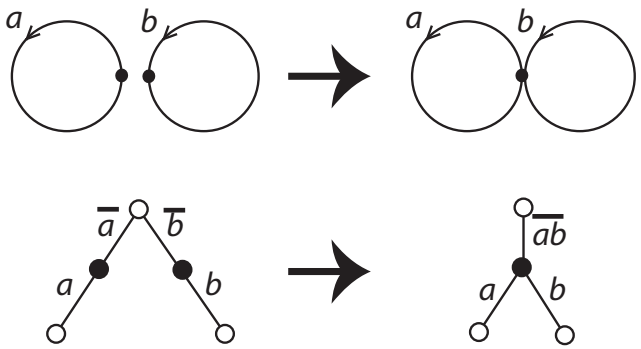

Fig. 9. Two regions merging at a point

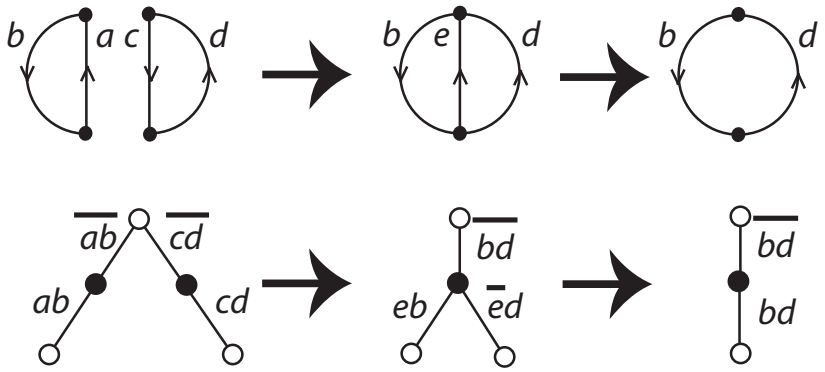

Fig. 10. Two regions merging at an edge and then the edge is deleted

new edge e. Strictly speaking, we should say that semi-edges a and $\overline{c}$ merge to semi-edge e, and semi-edges $\overline{a}$ and c merge to semi-edge $\overline{e}$.

The first transition of the bottom part of the figure shows the action on the corresponding maptrees. As with point merge, edge merges are represented by foldings and concatenations in the maptree. The merge between edges a and c is represented in the maptree as a fold about the maptree root with concatenation of $\overline{ab}$ with $\overline{cd}$. Now, some algebraic structure comes into play. The elements $a, b, c, d, e, \ldots$, and $\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \ldots$ generate a group, where the operation is concatenation, the identity is the empty word Λ, and the inverse of x is $\overline{x}$. In our case, bearing in mind the cycle structure, we have:

$$\overline{b}\overline{a}\overline{c}\overline{d} = \overline{b}\Lambda\overline{c} = \overline{b}\overline{c}$$

because $\overline{a} = c$ and so $\overline{a}\overline{c} = \Lambda$. Also, we substitute e for a in label eb and and $\overline{e}$ for c in label cd.

Because the labels on maptree edges are cycles rather than plain words, there are other allowable concatenations, and these will result in other possible transformations of the regions.

4.3 Edge deletion

Both types of merge in the preceding subsection were represented by folding two edges of the map tree together, where the fold was made at a white node. We now give an example of an edge deletion and its representation, which turns out to be a fold about a black node. Continuing on from the merger shown in figure 10, we now delete edge e, as shown in the second transition of the top portion of figure. The second transition of the bottom part of the figure shows the action on the corresponding maptrees. In this case, the edges labeled eb and $\overline{e}b$ fold into the single edge labeled bd. This label also follows the operations of edge group, as defined in the previous section, where:

$$be\overline{e}d = b\Lambda d = bd$$

5 Conclusions

This paper has developed a topological representation of closed surficial embeddings. We have demonstrated that the representation does indeed contain sufficient information to generate unique embeddings, up to homeomorphism of the embedding surface. We have also shown how a simple modification provides a representation for planar embeddings.

One strand of this research continues earlier work on spatial relationships (see, for example, [6, 3]). While RCC and the n-intersection methods are able to represent some level of topological detail of the relationship between two regions, the maptree goes further in dealing with multiple regions and providing a full topological representation.

The theory of topological change is a complex one, and was merely sketched out in section 4, using some illustrative examples. Further work, currently in progress, is developing a comprehensive theory of the role of maptree transformations. In particular, we are developing a classification of kinds of folds, as well as the underlying structure of the labels (a group under concatenation). This forms part of a larger mission, outlined in [9] to develop an event-based approach to dynamic geospatial phenomena.

6 Acknowledgments

This material is partly based upon work supported by the US National Science Foundation under grant number IIS-0916219.

References

1. O.P. Buneman. A grammar for the topological analysis of plane figures. In B. Meltzer and D. Michie, editors, *Machine Intelligence*, volume 5, pages 383–393. Elselvier, 1970.
2. J.R. Edmonds. A combinatorial representation for polyhedral surfaces. *Notices Amer. Math. Soc.*, 7:646, 1960.
3. M.J. Egenhofer and R.D. Franzosa. Point-set topological spatial relations. *International Journal of Geographical Information Systems*, 5(2):161–174, 1991.
4. J. Jiang and M. Nittel, S. Worboys. Qualitative change detection using sensor networks based on connectivity information. *GeoInformatica*, 15(2):305—328, 2011.
5. J. Jiang and M. Worboys. Event-based topology for dynamic planar areal objects. *International Journal of Geographical Information Science*, 23(1):33–60, 2009.
6. D. A. Randell, Z. Cui, and A. Cohn. A spatial logic based on regions and connection. In B. Nebel, C. Rich, and W. Swartout, editors, *KR'92. Principles of Knowledge Representation and Reasoning: Proceedings of the Third International Conference*, pages 165–176. Morgan Kaufmann, San Mateo, California, 1992.
7. J. Stell and M. Worboys. Relations between adjacency trees. *Journal of Theoretical Computer Science*, 412:4452–4468, 2011.
8. W.T. Tutte. What is a map? In *New Directions in the Theory of Graphs,*, pages 309–325. Academic Press, New York, 1973.
9. M.F. Worboys. Event-oriented approaches to geographic phenomena. *International Journal of Geographic Information Science*, 19(1):1–28, 2005.